Monomial Resolutions

Juliann Geraci

Outline

- Background
- Taylor Resolution
- Mapping Cones
- Eliahou Kevaire Resolution

Background

Multigrading

The polynomial ring $S = k[x_1, \ldots, x_n]$ is $\mathbb{N}^n$-graded by

$$\mathrm{mdeg}(x_i) = (0, \ldots, 0, 1_i, 0, \ldots, 0)$$

where mdeg stands for **multidegree**.

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We say that $x_1^{a_1} \cdots x_n^{a_n}$ has $\mathbb{N}^n$-degree $(a_1, \ldots, a_n)$ or multidegree $x_1^{a_1} \ldots x_n^{a_n}$.

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Any two monomials have a different multidegree.

Background

Definition

An ideal $M \subset S$ is a **monomial ideal** if it is generated by monomials.

- ♥ $(x^2, xy, y^3) \subset k[x, y]$
- ♥ $(x^2y, xz, xyz, yz) \subset k[x, y, z]$

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Definition

$S(x_1^{a_1} \cdots x_n^{a_n}) = S$ with 1 in multidegree $(x_1^{a_1} \cdots x_n^{a_n})$

Background

Motivation

Relations on (x^2, y^3)

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Relations on (x^2, y^3)

- $y^3 x^2 - x^2 y^3$

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Relations on (x^2, y^3)

- ♥ $y^3x^2 - x^2y^3$

Relations on (x^2, xy, y^3)

- ♥ $yx^2 - xxy$
- ♥ $y^2xy - xy^3$

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Definition

An **exterior algebra**, E, is a k-algebra generated by $e_1, \ldots, e_m$ with multiplication denoted $\wedge$, with relations

$$\begin{cases} e_1 \wedge e_j = -e_j \wedge e_i \\ e_i \wedge e_i = 0 \end{cases}$$

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- A k-vector space basis of E is $e_{j_1} \wedge \ldots e_{j_i}$
- $E = \bigoplus_{i-0}^{n} E_i$
- $\dim(E_i) = \binom{n}{i}$

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The **Koszul Complex K** on a set of polynomials $f_1, \ldots, f_m$ is built from an exterior algebra E over k on basis elements $e_1 \ldots e_m$.

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$$0 \to S \otimes_k E_m \to S \otimes_k E_{m-1} \to \ldots \to S \otimes_k E_1 \to S \otimes_k E_0 \to 0$$

as an S-module with differential

$$d(e_{j_1} \wedge \ldots e_{j_i}) = \sum_{1 \leq p \leq i} (-1)^{p+1} f_{j_p} (e_{j_1} \wedge \ldots \wedge \hat{e_{j_p}} \wedge \ldots \wedge e_{j_i})$$

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- ♥ $K(\underline{f})$ is exact if and only if $f_1, \ldots, f_m$ form a regular sequence.

Taylor's Resolution

Construction

Let M be a monomial ideal generated my monomials $m_1, \ldots, m_q$. Let E be the exterior algebra over k on basis elements $e_1, \ldots, e_q$. The complex $\mathbf{T}_M = S \otimes_k E$ as an S-module equipped with the differential

$$d(e_{j_1} \wedge \ldots \wedge e_{j_i}) = \sum_{1 \leq p \leq i} (-1)^{p-1} \frac{\operatorname{lcm}(m_{j_1}, \ldots m_{j_i})}{\operatorname{lcm}(m_{j_1}, \ldots \hat{m}_{j_p}, \ldots, m_{j_i})} e_{j_1} \wedge \ldots \wedge \hat{e_{j_p}} \wedge \ldots \wedge e_{j_i}$$

Taylor's Resolution

Example

Let $S = k[x, y]$ and $M = (x^2, xy, y^3)$. Let's resolve S/M over S !

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Let e_1, e_2, e_3 be standard basis elements in homological degree 1.

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Let $S = k[x, y]$ and $M = (x^2, xy, y^3)$. Let's resolve S/M over S !

Let e_1, e_2, e_3 be standard basis elements in homological degree 1.

$$\begin{matrix} S(x^2) \\ \bigoplus \\ S(xy) \\ \bigoplus \\ S(y^3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S \to 0$$

Taylor's Resolution

In homological degree 2, we have standard basis $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$.

$$d(e_1 \wedge e_2) = \frac{\operatorname{lcm}(x^2, xy)}{\operatorname{lcm}(xy)} e_2 - \frac{\operatorname{lcm}(x^2, xy)}{\operatorname{lcm}(x^2)} e_1 = xe_2 - ye_1$$

$$d(e_1 \wedge e_3) = \frac{\operatorname{lcm}(x^2, y^3)}{\operatorname{lcm}(x^2)} e_3 - \frac{\operatorname{lcm}(x^2, y^3)}{\operatorname{lcm}(y^3)} e_1 = y^3 e_3 - x^2 e_1$$

$$d(e_2 \wedge e_3) = \frac{\operatorname{lcm}(xy, y^3)}{\operatorname{lcm}(y^3)} e_3 - \frac{\operatorname{lcm}(xy, y^3)}{\operatorname{lcm}(xy)} e_2 = xe_3 - y^2 e_2$$

$$\begin{matrix} S(x^2y) \\ \oplus \\ S(x^2y^3) \\ \oplus \\ S(xy^3) \end{matrix} \xrightarrow{\begin{bmatrix} -y & -y^3 & 0 \\ x & 0 & -y^2 \\ 0 & x^2 & x \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(y^3) \\ \oplus \\ S(xy) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S \to 0$$

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This is **not** minimal.

Taylor's Resolution

Pro's	Con's
▪ simple structure	▪ highly non minimal
▪ very similar to the Koszul Complex	
▪ upper bound on the sum of the multidegrees of any free module in the resolution	
$\leq \deg(m_1) + \ldots + \deg(m_q) - q$	
▪ upper bound on Betti numbers	
$\beta_i \leq \binom{q}{i}$	

Mapping Cones

Definition

If $\mathbf{U}, \mathbf{U}'$ are resolutions for S/I and S/J and we have a map $S/I \to S/J$, then there exists $\varphi : \mathbf{U} \to \mathbf{U}'$, called the **comparison map**.

Mapping Cones

Definition

If $\mathbf{U}, \mathbf{U}'$ are resolutions for S/I and S/J and we have a map $S/I \to S/J$, then there exists $\varphi : \mathbf{U} \to \mathbf{U}'$, called the **comparison map**. **Goal:** Given a short exact sequence

$$0 \to S/I \to S/J \to S/L \to 0$$

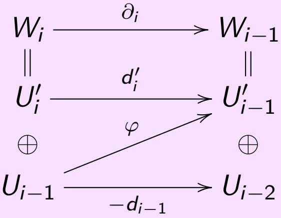
and a free resolution $\mathbf{U}$ for S/I and $\mathbf{U}'$; construct free resolution for S/L

Mapping Cones

Definition

The ***mapping cone of*** φ is the complex $\mathbf{W}$ with differential ∂ defined as follows:

$$W_i = U_{i-1} \oplus U'_i \text{ as a module}$$
$$\partial\,|_{U_{i-1}} = -d + \varphi : U_{i-1} \to U_{i-2} \oplus U'_{i-1}$$
$$\partial\,|_{U'_i} = d' : U'_i \to U'_{i-1}$$



Mapping Cones

Construction

Let M be an ideal minimally generated by monomials $m_1, \ldots, m_r$. Set $M_i = (m_1, \ldots m_i)$ for $1 \leq i \leq r$. For each $i \geq 1$, we have

$$0 \to S/(M_i : m_{i+1})(m_{i+1}) \xrightarrow{m_{i+1}} S/M_i \longrightarrow S/M_{i+1} \to 0.$$

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Given:

- ♥ multigraded free resolution $\mathbf{F}_i$ of S/M_i
- ♥ multigraded free resolution $\mathbf{G}_i$ of $S/(M_i : m_{i+1})$

Construct:

- ♥ multigraded free resolution $\mathbf{F}_{i+1}$ of S/M_{i+1}.

We say the multigraded free resolution $\mathbf{F}_q$ of S/M obtained in this way, is obtained by ***iterated mapping cones***.

Mapping Cones

Example

Let $S = k[x, y]$ and $M = (x^2, y^3, xy)$. Let's resolve S/M over S via iterated mapping cones!

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Let $S = k[x, y]$ and $M = (x^2, y^3, xy)$. Let's resolve S/M over S via iterated mapping cones!

We have

- ♥ $M_1 = (x^2)$
- ♥ $M_2 = (x^2, y^3)$
- ♥ $M_3 = M = (x^2, y^3, xy)$

Mapping Cones

The second short exact sequence is

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Minimal resolution of $S/(x, y^2)$

$$0 \to S(x^2y^3) \xrightarrow{\begin{bmatrix} y^2 \\ -x \end{bmatrix}} S(x^2y) \oplus S(xy^3) \xrightarrow{\begin{bmatrix} x & y^2 \end{bmatrix}} S(xy) \to \frac{S}{(x, y^2)}(xy) \to 0$$

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\begin{array}{ccccccccc}
0 & \longrightarrow & S(x^2y^3) & \xrightarrow{\begin{bmatrix} y^2 \\ -x \end{bmatrix}} & S(x^2y) \oplus S(xy^3) & \xrightarrow{\begin{bmatrix} x & y^2 \end{bmatrix}} & S(xy) & \longrightarrow & \frac{S}{(x,y^2)}(xy) & \longrightarrow & 0 \\
 & & \varphi \Big\downarrow 1 & & \varphi \Big\downarrow \begin{bmatrix} y & 0 \\ 0 & x \end{bmatrix} & & \varphi \Big\downarrow \cdot xy & & \varphi \Big\downarrow \cdot xy & & \\
0 & \longrightarrow & S(x^2y^3) & \xrightarrow{\begin{bmatrix} y^2 \\ -x \end{bmatrix}} & S(x^2) \oplus S(y^3) & \xrightarrow{\begin{bmatrix} x^2 & y^3 \end{bmatrix}} & S & \longrightarrow & \frac{S}{(x^2,y^3)} & \longrightarrow & 0
\end{array}
$$

Mapping Cones

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\begin{array}{ccccccccc}
 & & U_2' & & U_1' & & U_0' & & \\
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Mapping Cones

$$0 \to S(x^2y^3) \xrightarrow{\begin{bmatrix} 1 \\ y^2 \\ -x \end{bmatrix}} \begin{matrix} S(x^2y^3) \\ \oplus \\ S(x^2y) \\ \oplus \\ S(xy^2) \end{matrix} \xrightarrow{\begin{bmatrix} y^3 & y & 0 \\ -x^2 & 0 & x \\ 0 & -x & -y^2 \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(y^3) \\ \oplus \\ S(xy) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & y^3 & xy \end{bmatrix}} S \to \frac{S}{(x^2, xy, y^3)}$$

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Eliahou-Kevaire Resolutions

Definition

A monomial ideal $M \subseteq S = k[x_1, \ldots, x_n]$is **Borel** if it satisies the *Borel property*: if $i < j$, g a monomial such that $gx_j \in M$, then $gx_i \in M$.

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- ♥ (x, y, z)
- ♥ (x^2, xy, y^3)
- ♥ (x^2, xy, xz)

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- max(m)=max$\{i \mid x_i \text{ divides } m\}$
- min(m)=min$\{i \mid x_i \text{ divides } m\}$

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- max(m)=$\max\{i \mid x_i \text{ divides } m\}$
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Every monomial w of Borel ideal M admits a unique decomposition$w = uv$ such that u is a minimal monomial generator of M and max(u) $\leq$min(v).

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Every monomial w of Borel ideal M admits a unique decomposition$w = uv$ such that u is a minimal monomial generator of M and max$(u) \leq$min(v).
Denote beginning of $w = b(w) = u$ and end of $w = e(w) = v$.

- $x^2yz^3 \in (x^2, z)$
- $x^2yz^3 = (x^2)(yz^3)$

Eliahou-Kevaire Resolutions

Construction

Let M be a Borel ideal with minimal monomial generators $m_1, \ldots, m_r$. For each m_i and for each sequence $1 \leq j_i < \ldots < j_p \leq \max(m_i)$ of strictly increasing natural numbers we conosider the free S-module $S(m_i x_{j_1} \cdots x_{j_p})$ with one generator, denoted $(mi; j_1, \ldots, j_p)$ in homological degree $p+1$ and multidegree $m_i x_{j_1} \cdots x_{j_p}$.

Definition

The **Eliahou-Kevaire resolution $\mathbf{E}_M$** of S/M has basis

$$\mathcal{B} = \{1\} \cup \{(m_i; j_1, \ldots, j_p) \mid 1 \leq j_i < \ldots < j_p \leq \max(m_i), 1 \leq i \leq r\}$$

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1 is a basis of S in homological degree 0

$(m_1; \emptyset) \ldots (m_r; \emptyset)$ is the basis in homological degree 1

Eliahou-Kevaire Resolutions

Define maps

$$\partial(m_i; j_1, \ldots, j_p) = \sum_{q=1}^{p} (-1)^q x_{j_q} (m_i; j_1, \ldots, \hat{j_q} \ldots, j_p)$$

$$\mu(m_i; j_1, \ldots, j_p) = \sum_{q=1}^{p} (-1)^q \frac{m_i x_{j_q}}{b(m_i x_{j_q})} (b(m_i x_{j_q}); j_1, \ldots, \hat{j_q} \ldots, j_p)$$

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The differential in E_M is $d = \partial - \mu$

Eliahou-Kevaire Resolutions

Example

Let $S = k[x, y]$ and $M = (x^2, y^3, xy)$. Let's resolve S/M over S using the Eloiahou-Kevaire Resolution!

Eliahou-Kevaire Resolutions

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Basis	Homological degree
1	0
$(x^2;\emptyset), (xy;\emptyset), (y^3;\emptyset)$	1
$(xy;1), (y^3;1)$	2

Eliahou-Kevaire Resolutions

$$\begin{matrix} S(x^2) \\ \bigoplus \\ S(xy) \\ \bigoplus \\ S(y^3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

Eliahou-Kevaire Resolutions

$$\begin{matrix} S(x^2) \\ \bigoplus \\ S(xy) \\ \bigoplus \\ S(y^3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

$$\partial(xy;1) = (-1)^1 x \cdot (xy;\emptyset)$$

$$\mu(xy;1) = (-1)^1 \frac{xy \cdot x}{b(xy \cdot x)}(b(xy \cdot x);\emptyset) = -y \cdot (x^2;\emptyset)$$

$$\partial(y^3;1) = (-1)^1 x \cdot (y^3;\emptyset)$$

$$\mu(y^3;1) = (-1)^1 \frac{y^3 \cdot x}{b(y^3 \cdot x)}(b(y^3 \cdot x);\emptyset) = -y^2 \cdot (xy;\emptyset)$$

Eliahou-Kevaire Resolutions

$$0 \longrightarrow \begin{matrix} S(x^2y) \\ \oplus \\ S(xy^3) \end{matrix} \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(xy) \\ \oplus \\ S(y^3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

Eliahou-Kevaire Resolutions

Pro's	Con's
▪ always minimal	▪ only works with Borel ideals
▪ it's an iterated mapping cone	
▪ know ALOT of numerical invariants	

Eliahou-Kevaire Resolution

- $\text{height}(M) = \max\{j \mid \text{a power of } x_j \text{ in} M\}$
- $\text{reg}(M)$=highest degree of a monomial generator of M
- $\text{pd}(M) = \max\{\max(m_i) - 1 \mid 1 \leq i \leq r\}$
- $b_p^S(M) = \sum_{i=1}^{r} \binom{\max(m_i) - 1}{p}$
- $b_{p,p+q}^S(M) = \sum_{\substack{\deg(m_i)=q \\ 1 \leq i \leq r}}^{r} \binom{\max(m_i) - 1}{p}$

Comparing Resolutions

$$0 \to S(x^2y^3) \xrightarrow{\begin{bmatrix} y^2 \\ -1 \\ x \end{bmatrix}} \begin{matrix} S(x^2y) \\ \oplus \\ S(x^2y^3) \\ \oplus \\ S(xy^3) \end{matrix} \xrightarrow{\begin{bmatrix} -y & -y^3 & 0 \\ x & 0 & -y^2 \\ 0 & x^2 & x \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(y^3) \\ \oplus \\ S(xy) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

Comparing Resolutions

$$0 \to S(x^2y^3) \xrightarrow{\begin{bmatrix} y^2 \\ -1 \\ x \end{bmatrix}} \begin{matrix} S(x^2y) \\ \oplus \\ S(x^2y^3) \\ \oplus \\ S(xy^3) \end{matrix} \xrightarrow{\begin{bmatrix} -y & -y^3 & 0 \\ x & 0 & -y^2 \\ 0 & x^2 & x \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(y^3) \\ \oplus \\ S(xy) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

We can prune the above Taylor Resolution to get...

Comparing Resolutions

$$0 \longrightarrow S(x^2y^3) \xrightarrow{\begin{bmatrix} -1 \end{bmatrix}} S(x^2y^3)$$

$$\oplus \qquad \oplus$$

$$0 \longrightarrow \begin{matrix} S(x^2y) \\ \oplus \\ S(xy^3) \end{matrix} \xrightarrow{\begin{bmatrix} y & 0 \\ -x & y^2 \\ 0 & -x \end{bmatrix}} \begin{matrix} S(x^2) \\ \oplus \\ S(xy) \\ \oplus \\ S(y^3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy & y^3 \end{bmatrix}} S$$

The Eliahou-Kevaire Resolution as a minimal resolution of S/M direct sum with some extra bits.