A New Explicit Finite Free Resolution of Ideals Generated by Monomials in an R-sequence

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Motivation

- How can we describe the structure of a module?

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$$\dots A_2 \xrightarrow{d_2=\begin{pmatrix}\text{relations on}\\ \text{the relations}\\ \text{in } d_1\end{pmatrix}} A_1 \xrightarrow{d_1=\begin{pmatrix}\text{relations on the}\\ \text{generators of } M\end{pmatrix}} A_0 \xrightarrow{\begin{pmatrix}\text{generators}\\ \text{of } M\end{pmatrix}} M \to 0$$

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  - Hilbert series
  - Betti Numbers
  - Ext, Tor and Hom functors

Preliminary Definitions

Let A, B, C be R-modules.

Definition

The pair of homomorphisms $A \xrightarrow{\psi} B \xrightarrow{\varphi} C$ is said to be **exact** (at B) if $\operatorname{im} \psi = \ker \varphi$.

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A sequence $\dots A_{n-1} \to A_n \to A_{n+1} \dots$ of homomorphisms is said to be an **exact sequence** if it is exact at every A_n between a pair of homomorphisms.

Preliminary Definitions

Definition
Given a module M over a ring R, a **free resolution** of M is an exact sequence of free R-modules

$$\dots \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} \dots \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{\varepsilon} M \to 0$$

where d_i are homomorphisms called **differentials** and ε is called the **augmentation map**

Finding a (close to) Minimal Resolution

The best case scenario: construct a minimal free resolution.

- A free resolution is minimal if and only if at each step we make an optimal choice, that is, we choose a minimal system of generators of the kernel in order to construct the next differential.
- A minimal free resolution is smallest in the sense that it lies (as a direct summand) inside any other free resolution of the module.

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Two particularly "nice" resolutions that are close to minimal

- Taylor Resolution
- Lyubeznik Resolution

Preamble

Let K be a field and let $R = K[x_0, \ldots, x_n]$.

Let $y_1, \ldots, y_m$ be m monomials in the x_i, with $\mathcal{I} = (y_1, \ldots, y_m)$, an ideal of R.

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Some examples of monomial ideals;

- For $R = \mathbb{C}[x, y, z]$
  - (x, y, z)
  - (xy, z^2)
  - $(x^y z^3, x)$

The Taylor Resolution

The **Taylor Resolution** of $R/\mathcal{I}$ is given as

$$0 \to R^{\binom{m}{m}} \to \ldots \to R^{\binom{m}{s}} \to R^{\binom{m}{s-1}} \to \ldots \to R^{\binom{m}{1}} \to R \to R/\mathcal{I} \to 0$$

The differential; for $I = \{i_1, \ldots, i_s\}$,

$$d(e_I) = \sum_{j=1}^{j=s} (-1)^{j+1} \frac{\operatorname{lcm}(y_{i_1}, \ldots, y_{i_s})}{\operatorname{lcm}(y_{i_1}, \ldots, \hat{y}_{i_j}, \ldots y_{i_s})} e_{I \setminus i_j}$$

The Taylor Resolution

Example: Let $R = K[a, b, c]$ and $\mathcal{I} = (a^2, ab, b^3)$. Then the Taylor resolution of $R/\mathcal{I}$ is

$$0 \to R \xrightarrow{\begin{pmatrix} a \\ -1 \\ b^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & -b^3 & -b \\ -b^2 & 0 & a \\ a & a^2 & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} R \to R/\mathcal{I} \to 0$$

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The Lyubeznik Resolution

The **Lyubeznik Resolution** of $R/\mathcal{I}$ is a subcomplex of the Taylor resolution, which is generated in homological desgree s by the basis elements $e_{i_1,\ldots,i_s}(1 \leq i_1 < \ldots i_s \leq m)$ such that for every $t < s$, $q < i_t$: $y_q \not| \operatorname{lcm}(y_{i_t}, \ldots, y_{i_s})$.

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The differential is the same as the Taylor resolution, as it is a subcomplex. For $I = \{i_1, \ldots, i_s\}$,

$$d(e_I) = \sum_{j=1}^{j=s} (-1)^{j+1} \frac{\operatorname{lcm}(y_{i_1}, \ldots, y_{i_s})}{\operatorname{lcm}(y_{i_1}, \ldots, \hat{y_{i_j}}, \ldots y_{i_s})} e_{I \setminus i_j}$$

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Comparison

$$\mathbf{T}: 0 \to R \xrightarrow{\begin{pmatrix} a \\ -1 \\ b^2 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & -b^3 & -b \\ -b^2 & 0 & a \\ a & a^2 & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} R \to R/\mathcal{I} \to 0$$

Rank; 8

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Could this be minimal?

Comparison

Using Macaulay2, we can compute the minimal resolution.

```
i1 : R=QQ[a,b]

o1 = R

o1 : PolynomialRing

i2 : I=ideal(a^2,a*b,b^3)

             2        3
o2 = ideal (a , a*b, b )

o2 : Ideal of R

i3 : C= res I

      1      3      2
o3 = R  <-- R  <-- R  <-- 0

     0      1      2      3

o3 : ChainComplex

i4 : C.dd

          1                 3
o4 = 0 : R  <---------------- R  : 1
               | a2 ab b3 |

          3                   2
     1 : R  <------------------ R  : 2
               {2} | -b 0   |
               {2} | a  -b2 |
               {3} | 0  a   |

          2
     2 : R  <----- 0 : 3
               0

o4 : ChainComplexMap
```

That’s familiar!

References and Acknowledgements

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